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# Building an AdS/CFT superconductor

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## Abstract

We show that a simple gravitational theory can provide a holographically dual description of a superconductor. There is a critical temperature, below which a charged condensate forms via a second order phase transition and the (DC) conductivity becomes infinite. The frequency dependent conductivity develops a gap determined by the condensate. We find evidence that the condensate consists of pairs of quasiparticles.

# 1 Introduction

A most remarkable result to emerge from string theory is the AdS/CFT correspondence [1], which relates string theory on asymptotically anti de Sitter spacetimes to a conformal field theory on the boundary. In recent years, it has become clear that this holographic correspondence between a gravitational theory and a quantum field theory can be extended to describe aspects of nuclear physics such as the results of heavy ion collisions at RHIC [2] and to certain condensed matter systems. Phenomena such as the Hall effect [3] and Nernst effect [4, 5, 6] have dual gravitational descriptions. One can ask if there is a dual gravitational description of superconductivity.

Conventional superconductors, including many metallic elements (Al, Nb, Pb, ...), are well described by BCS theory [7]. However, basic aspects of unconventional superconductors, including the pairing mechanism, remain incompletely understood. There are many indications that the normal state in these materials is not described by the standard Fermi liquid theory [8]. We therefore hope that a tractable theoretical model of a strongly coupled system which develops superconductivity will be of interest. Several important unconventional superconductors, such as the cuprates and organics, are layered and much of the physics is 2+1 dimensional. Our model will also be 2+1 dimensional.

To map a superconductor to a gravity dual, we introduce temperature by adding a black hole [9] and a condensate through a charged scalar field. To reproduce the superconductor phase diagram, we require a system that admits black holes with scalar hair at low temperature, but no hair at high temperature. While Hertog has shown that neutral AdS black holes can have neutral scalar hair only if the theory is unstable [10], Gubser has recently suggested that a charged black hole will support charged scalar hair if the charges are large enough [11]. We consider a simpler version of Gubser's bulk theory (in which the black hole can remain neutral) and show that it indeed provides a dual description of a superconductor.

## 2 The model: condensing charged operators

We start with the planar Schwarzschild anti-de Sitter black hole

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2 + dy^2), \quad (1)$$

where

$$f = \frac{r^2}{L^2} - \frac{M}{r}. \quad (2)$$

$L$  is the AdS radius and  $M$  determines the Hawking temperature of the black hole:

$$T = \frac{3M^{1/3}}{4\pi L^{4/3}}. \quad (3)$$

This black hole is 3+1 dimensional, and so will be dual to a 2+1 dimensional theory. In this background, we now consider a Maxwell field and a charged complex scalar field, with Lagrangian density<sup>1</sup>

$$\mathcal{L} = -\frac{1}{4}F^{ab}F_{ab} - V(|\Psi|) - |\partial\Psi - iA\Psi|^2. \quad (4)$$

For simplicity and concreteness, we will focus on the case

$$V(|\Psi|) = -\frac{2|\Psi|^2}{L^2}. \quad (5)$$

Although the mass squared is negative, it is above the Breitenlohner-Freedman bound [12] and hence does not induce an instability. It corresponds to a conformally coupled scalar in our background (1) and arises in several contexts in which the  $AdS_4/CFT_3$  correspondence is embedded into string theory. For instance, the truncation of M theory on  $AdS_4 \times S^7$  to  $\mathcal{N} = 8$  gauged supergravity has scalars and pseudoscalars with this mass, dual to the bilinear operators  $\text{tr}\Phi^{(I}\Phi^{J)}$  and  $\text{tr}\Psi^{(I}\Psi^{J)}$  in the dual  $\mathcal{N} = 8$  Super Yang-Mills theory, respectively. However, we should note that our Lagrangian (4) has not been obtained from M theory. We expect that our choice of mass is not crucial, and qualitatively similar results will hold, e.g., for massless fields.

We will work in a limit in which the Maxwell field and scalar field do not backreact on the metric. This limit is consistent as long as the fields are small in Planck units. (Recall that in the analogous case one dimension higher, only Yang-Mills states with energy of order  $N^2$  have finite backreaction in the bulk.) Alternatively, this decoupled Abelian-Higgs sector can be obtained from the full Einstein-Maxwell-scalar theory considered in [11] through a scaling limit in which the product of the charge of the black hole and the charge of the scalar field is held fixed while the latter is taken to infinity. Thus we will obtain solutions of non-backreacting scalar hair on the black hole. As we shall see, our simple model captures the physics of interest.

Taking a plane symmetric ansatz,  $\Psi = \Psi(r)$ , the scalar field equation of motion is

$$\Psi'' + \left(\frac{f'}{f} + \frac{2}{r}\right)\Psi' + \frac{\Phi^2}{f^2}\Psi + \frac{2}{L^2f}\Psi = 0, \quad (6)$$

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<sup>1</sup>Introducing a gauge coupling  $1/e^2$  in front of the  $|F|^2$  term in the action is equivalent to rescaling the fields  $\Psi \rightarrow e\Psi$  and  $A_\mu \rightarrow eA_\mu$ . Setting  $e = 1$  is a choice of units of charge in the dual 2+1 theory.

where the scalar potential  $A_t = \Phi$ . With  $A_r = A_x = A_y = 0$ , the Maxwell equations imply that the phase of  $\Psi$  must be constant. Without loss of generality we therefore take  $\Psi$  to be real. The equation for the scalar potential  $\Phi$  is the time component of the equation of motion for a massive vector field

$$\Phi'' + \frac{2}{r}\Phi' - \frac{2\Psi^2}{f}\Phi = 0, \quad (7)$$

where  $2\Psi^2$  is the, in our case,  $r$  dependent mass. The charged condensate has triggered a Higgs mechanism in the bulk theory. At the horizon,  $r = r_0$ , for  $\Phi dt$  to have finite norm,  $\Phi = 0$ , and (6) then implies  $\Psi = -3r_0\Psi'/2$ . Thus, there is a two parameter family of solutions which are regular at the horizon. Integrating out to infinity, these solutions behave as

$$\Psi = \frac{\Psi^{(1)}}{r} + \frac{\Psi^{(2)}}{r^2} + \dots \quad (8)$$

and

$$\Phi = \mu - \frac{\rho}{r} + \dots \quad (9)$$

For  $\Psi$ , both of these falloffs are normalizable [13], so one can impose the boundary condition that either one vanishes.<sup>2</sup> After imposing the condition that either  $\Psi^{(1)}$  or  $\Psi^{(2)}$  vanish we have a one parameter family of solutions.

It follows from (7) that the solution for  $\Phi$  is always monotonic: It starts at zero and cannot have a positive maximum or a negative minimum. Note that even though the field equations are nonlinear, the overall signs of  $\Phi$  and  $\Psi$  are not fixed. We will take  $\Phi$  to be positive and hence have a system with positive charge density. The sign of  $\Psi$  is part of the freedom to choose the overall phase of  $\Psi$ .

Properties of the dual field theory can be read off from the asymptotic behavior of the solution. For example, the asymptotic behavior (9) of  $\Phi$  yields the chemical potential  $\mu$  and charge density  $\rho$  of the field theory. The condensate of the scalar operator  $\mathcal{O}$  in the field theory dual to the field  $\Psi$  is given by

$$\langle \mathcal{O}_i \rangle = \sqrt{2}\Psi^{(i)}, \quad i = 1, 2 \quad (10)$$

with the boundary condition  $\epsilon_{ij}\Psi^{(j)} = 0$ . The  $\sqrt{2}$  normalization simplifies subsequent formulae, and corresponds to taking the bulk-boundary coupling  $\frac{1}{2}\int d^3x (\bar{\mathcal{O}}\Psi + \mathcal{O}\bar{\Psi})$ . Note that  $\mathcal{O}_i$  is an operator with dimension  $i$ . From this point on we will work in units in which

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<sup>2</sup>One might also imagine imposing boundary conditions in which both  $\Psi^{(1)}$  and  $\Psi^{(2)}$  are nonzero. However, if these boundary conditions respect the AdS symmetries, then the result is a theory in which the asymptotic AdS region is unstable [14].

the AdS radius is  $L = 1$ . Recall that  $T$  has mass dimension one, and  $\rho$  has mass dimension two so  $\langle \mathcal{O}_i \rangle / T^i$  and  $\rho / T^2$  are dimensionless quantities.

An exact solution to eqs (6,7) is clearly  $\Psi = 0$  and  $\Phi = \mu - \rho/r$ . It appears difficult to find other analytic solutions to these nonlinear equations. However, it is straightforward to solve them numerically. We find that solutions exist with all values of the condensate  $\langle \mathcal{O} \rangle$ . However, as shown in figure 1, in order for the operator to condense, a minimal ratio of charge density over temperature squared is required.

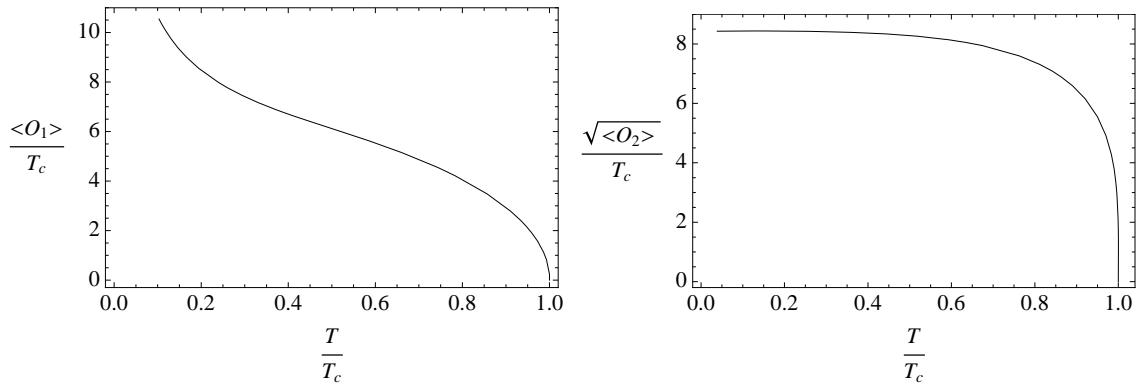


Figure 1: The condensate as a function of temperature for the two operators  $\mathcal{O}_1$  and  $\mathcal{O}_2$ . The condensate goes to zero at  $T = T_c \propto \rho^{1/2}$ .

The right hand curve in figure 1 is qualitatively similar to that obtained in BCS theory, and observed in many materials, where the condensate goes to a constant at zero temperature. The left hand curve starts similarly, but at low temperature the condensate appears to diverge as  $T^{-1/6}$ . However, when the condensate becomes very large, the backreaction on the bulk metric can no longer be neglected. At extremely low temperatures, we will eventually be outside the region of validity of our approximation.

By fitting these curves, we see that for small condensate there is a square root behaviour that is typical of second order phase transitions. Specifically, for one boundary condition we find

$$\langle \mathcal{O}_1 \rangle \approx 9.3 T_c (1 - T/T_c)^{1/2}, \quad \text{as } T \rightarrow T_c, \quad (11)$$

where the critical temperature is  $T_c \approx 0.226\rho^{1/2}$ . For the other boundary condition

$$\langle \mathcal{O}_2 \rangle \approx 144 T_c^2 (1 - T/T_c)^{1/2}, \quad \text{as } T \rightarrow T_c, \quad (12)$$

where now  $T_c \approx 0.118\rho^{1/2}$ . The continuity of the transition can be checked by computing the free energy. Finite temperature continuous symmetry breaking phase transitions are

only possible in 2+1 dimensions in the large  $N$  limit (i.e. the classical gravity limit of our model), where fluctuations are suppressed. These transitions will become crossovers at finite  $N$ .

Thus for  $T < T_c$  a charged scalar operator,  $\langle \mathcal{O}_1 \rangle$  or  $\langle \mathcal{O}_2 \rangle$ , has condensed. It is natural to expect that this condensate will lead to superconductivity of the current associated with this charge.

### 3 Maxwell perturbations and the conductivity

We now compute the conductivity in the dual CFT as a function of frequency. As a first step, we need to solve for fluctuations of the vector potential  $A_x$  in the bulk. The Maxwell equation at zero spatial momentum and with a time dependence of the form  $e^{-i\omega t}$  gives

$$A''_x + \frac{f'}{f} A'_x + \left( \frac{\omega^2}{f^2} - \frac{2\Psi^2}{f} \right) A_x = 0. \quad (13)$$

To compute causal behavior, we solve this equation with ingoing wave boundary conditions at the horizon [16]:  $A_x \propto f^{-i\omega/3r_0}$ . The asymptotic behaviour of the Maxwell field at large radius is seen to be

$$A_x = A_x^{(0)} + \frac{A_x^{(1)}}{r} + \dots \quad (14)$$

The AdS/CFT dictionary tells us that the dual source and expectation value for the current are given by

$$A_x = A_x^{(0)}, \quad \langle J_x \rangle = A_x^{(1)}. \quad (15)$$

Now from Ohm's law we can obtain the conductivity

$$\sigma(\omega) = \frac{\langle J_x \rangle}{E_x} = -\frac{\langle J_x \rangle}{\dot{A}_x} = -\frac{i\langle J_x \rangle}{\omega A_x} = -\frac{iA_x^{(1)}}{\omega A_x^{(0)}}. \quad (16)$$

In figure 2 we plot the frequency dependent conductivity obtained by solving (13) numerically. The horizontal line corresponds to temperatures at or above the critical value, where there is no condensate. The fact that the conductivity in the normal phase is frequency independent is a characteristic of theories with  $AdS_4$  duals [15]. The subsequent curves describe successively lower values of the temperature (for fixed charge density). We see that as the temperature is lowered, a gap opens. The gap becomes increasingly deep until the (real part of the) conductivity is exponentially small.

There is also a delta function at  $\omega = 0$  which appears as soon as  $T < T_c$ . This can be seen by looking at the imaginary part of the conductivity. The Kramers-Kronig relations

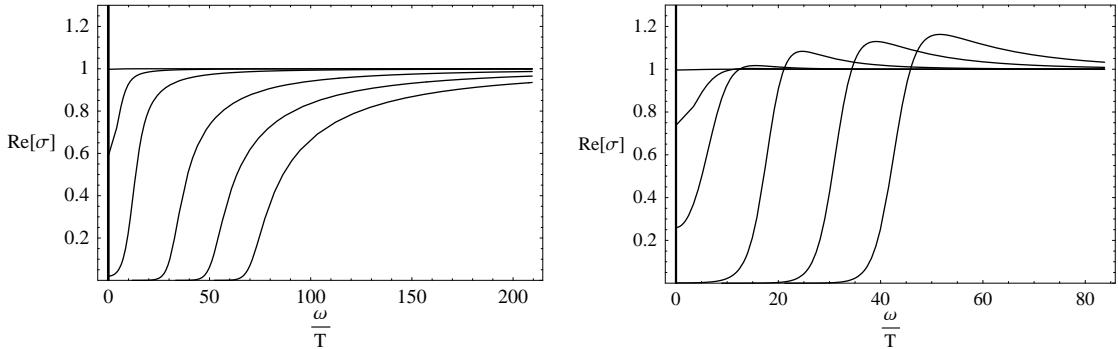


Figure 2: The formation of a gap in the real, dissipative, part of the conductivity as the temperature is lowered below the critical temperature. Results shown for both the  $\mathcal{O}_1$  operator (left) and the  $\mathcal{O}_2$  operator (right). There is also a delta function at  $\omega = 0$ . The rightmost curve in each plot corresponds to  $T/T_c = .0066$  (left) and  $T/T_c = .0026$  (right).

relate the real and imaginary parts of any causal quantity, such as the conductivity, when expressed in frequency space. They may be derived from contour integration and the fact that functions that vanish when  $t < 0$ , when transformed to frequency space only have poles in the lower half plane (in our conventions). One of the relations is

$$\text{Im}[\sigma(\omega)] = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\text{Re}[\sigma(\omega')] d\omega'}{\omega' - \omega}. \quad (17)$$

From this formula we can see that the real part of the conductivity contains a delta function,  $\text{Re}[\sigma(\omega)] = \pi\delta(\omega)$ , if and only if the imaginary part has a pole,  $\text{Im}[\sigma(\omega)] = 1/\omega$ . One finds that there is indeed a pole in  $\text{Im}[\sigma]$  at  $\omega = 0$  for all  $T < T_c$ . The superfluid density is the coefficient of the delta function of the real part of the conductivity<sup>3</sup>

$$\text{Re}[\sigma(\omega)] \sim \pi n_s \delta(\omega). \quad (18)$$

By (17),  $n_s$  is also the coefficient of the pole in the imaginary part  $\text{Im}[\sigma(\omega)] \sim n_s/\omega$  as  $\omega \rightarrow 0$ . We find that the superfluid density vanishes linearly with  $T_c - T$ :

$$n_s \approx C_i (T_c - T) \quad \text{as} \quad T \rightarrow T_c, \quad (19)$$

where  $C_1 = 16.5$  for the  $\mathcal{O}_1$  theory while  $C_2 = 24$  for the  $\mathcal{O}_2$  theory.

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<sup>3</sup>The superfluid density is usually defined as the coefficient of  $\delta(\omega)$  multiplied by the mass of the electron. In simple two fluid models, this density is related to the London magnetic penetration depth,  $n_s = 1/4\pi\lambda_L^2$ . Our scaling (19) thus implies  $\lambda_L \sim (T_c - T)^{-1/2}$ , consistent with Landau-Ginzburg theory.

In figure 3 we rescaled the small  $T/T_c$  plots of figure 2 by plotting the frequency in units of the condensate rather than the temperature. The curves tend to a limit in which the width of the gap is proportional to the size of the condensate. The differing shapes of the plots in figure 3 are precisely what is expected from type II and type I coherence factors, respectively [7]. Type II coherence suppresses absorption near the edge of the gap, explaining the slower rise of  $\text{Re}[\sigma]$  in the left hand plot. It is possible that this difference is due to the operator  $\mathcal{O}_1$  being a pair of bosons and  $\mathcal{O}_2$  a pair of fermions, as in the case of  $AdS_4 \times S^7$ .

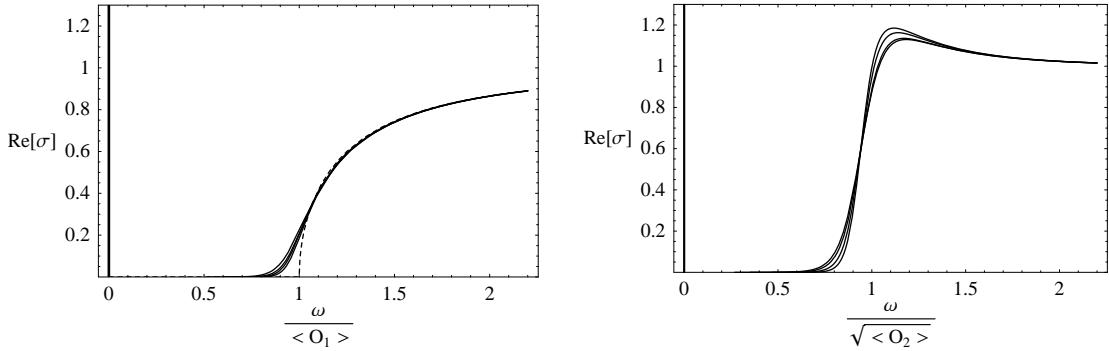


Figure 3: The gap at small  $T/T_c$ , with the frequency normalised in terms of the condensate. On the right the gap is finite, but since  $\langle \mathcal{O}_1 \rangle$  becomes large at small  $T$ , the gap on the left is also becoming large. The dashed curve on the left plot is (22).

The Ferrell-Glover sum rule states that  $\int \text{Re}[\sigma] d\omega$  is a constant independent of temperature. Thus the area missing under the curve  $\text{Re}[\sigma]$  due to the gap must be made up by the delta function at  $\omega = 0$ . That  $\text{Re}[\sigma]$  exceeds the value one in figure 3 (right) implies then that the superfluid density  $n_s$  must be correspondingly reduced for the  $\mathcal{O}_2$  system compared with the  $\mathcal{O}_1$  system for  $T \ll T_c$ .

We can also compute the contribution of the normal, non-superconducting, component to the DC conductivity. Let us define

$$n_n = \lim_{\omega \rightarrow 0} \text{Re}[\sigma(\omega)]. \quad (20)$$

From our numerics we obtain

$$n_n \sim e^{-\Delta/T}, \quad \text{for } \frac{\Delta}{T} \gg 1, \quad (21)$$

where we have  $\Delta = \langle \mathcal{O}_1 \rangle / 2$  and  $\Delta = \sqrt{\langle \mathcal{O}_2 \rangle} / 2$ . Numerically this factor of 1/2 is accurate to at least 4%. From (21),  $\Delta$  is immediately interpreted as the energy gap for charged

excitations. The gap we found previously in the frequency dependent conductivity was  $2\Delta$ . The extra factor of two is expected if the gapped charged quasiparticles are produced in pairs, suggesting that there is a ‘pairing mechanism’ at work in our model. Our results for  $\Delta$  are suggestive of strong pairing interactions. Note that for figure 1 (right) at  $T = 0$  we find  $2\Delta \approx 8.4T_c$ , which might be compared with the BCS prediction  $2\Delta \approx 3.54T_c$ . The larger value is what one expects for deeply bound Cooper pairs. Indeed, in our other model, figure 1 (left), we see that  $\Delta$  actually diverges at low  $T$ .

Finally in figure 4 we plot the imaginary part of the conductivity in this limiting, low temperature, regime. Again we see that the curves rapidly approach a limiting curve. We also see the advertised pole at  $\omega = 0$ .

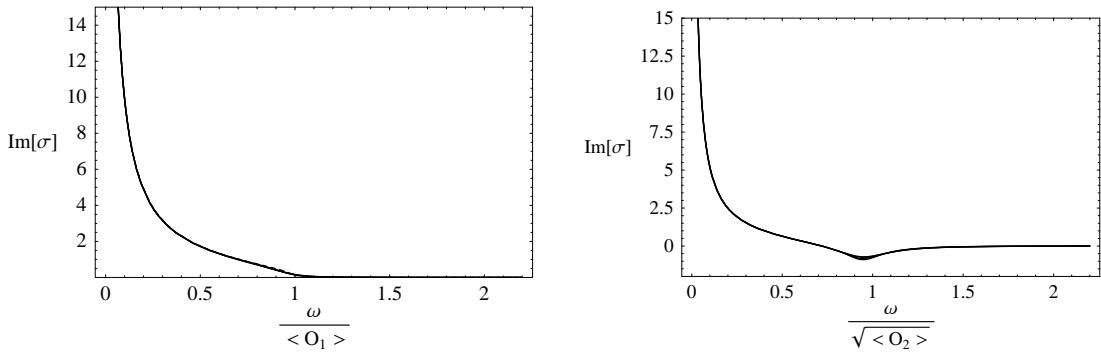


Figure 4: The imaginary part of the conductivity at small  $T/T_c$ , with the frequency normalised in terms of the condensate. The analytic expression (22) is also shown on the left, but is indistinguishable from the numerics.

It is natural to ask if one can reproduce this limiting low temperature behavior by just taking  $M = 0$  in our background metric, and considering our matter fields in anti de Sitter space. One problem is that there are no solutions to the field equations (6,7) which are smooth on the horizon of the Poincare patch. Nevertheless, for the  $\mathcal{O}_1$  case, we have observed numerically that at low temperatures,  $\Psi \approx \langle \mathcal{O}_1 \rangle / \sqrt{2}r$ . Taking  $M \rightarrow 0$  where  $f \approx r^2$ , (13) can be solved exactly to yield  $A_x = A_x^{(0)} \exp(\pm \sqrt{\langle \mathcal{O}_1 \rangle^2 - \omega^2}/r)$ . This exact result then produces the nonzero conductivities

$$\text{Re}[\sigma] = \frac{\sqrt{\omega^2 - \langle \mathcal{O}_1 \rangle^2}}{\omega} \quad \text{for } \omega > \langle \mathcal{O}_1 \rangle, \quad \text{Im}[\sigma] = \frac{\sqrt{\langle \mathcal{O}_1 \rangle^2 - \omega^2}}{\omega} \quad \text{for } \omega < \langle \mathcal{O}_1 \rangle, \quad (22)$$

via (16). The curves on the left hand sides of figures 3 and 4 are well approximated by the conductivity (22). We have included (22) as a dashed curve in these plots.

## 4 Discussion

We have shown that a simple 3+1 dimensional bulk theory can reproduce several properties of a 2+1 dimensional superconductor. Below a second order superconducting phase transition the DC superconductivity becomes infinite and an energy gap for charged excitations is formed.

There are many extensions of this model that we hope to consider elsewhere: 1) By probing the system with spatially varying fields and an external magnetic field, one can compute the superconducting coherence length and penetration depth, respectively. 2) One would like to consider a wider class of models by allowing for more general masses for the charged scalar field. 3) One should study the effects of backreaction on the bulk spacetime metric. 4) Perhaps the most interesting question is to understand the ‘pairing mechanism’ in field theory that leads to a condensate in these systems.

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## References

- [1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [[arXiv:hep-th/9711200](#)].
- [2] D. Mateos, “String Theory and Quantum Chromodynamics,” *Class. Quant. Grav.* **24**, S713 (2007) [[arXiv:0709.1523 \[hep-th\]](#)].
- [3] S. A. Hartnoll and P. Kovtun, “Hall conductivity from dyonic black holes,” *Phys. Rev. D* **76**, 066001 (2007) [[arXiv:0704.1160 \[hep-th\]](#)].
- [4] S. A. Hartnoll, P. K. Kovtun, M. Muller and S. Sachdev, “Theory of the Nernst effect near quantum phase transitions in condensed matter, and in dyonic black holes,” *Phys. Rev. B* **76**, 144502 (2007) [[arXiv:0706.3215 \[cond-mat.str-el\]](#)].
- [5] S. A. Hartnoll and C. P. Herzog, “Ohm’s Law at strong coupling: S duality and the cyclotron resonance,” *Phys. Rev. D* **76**, 106012 (2007) [[arXiv:0706.3228 \[hep-th\]](#)].

- [6] S. A. Hartnoll and C. P. Herzog, “Impure AdS/CFT,” arXiv:0801.1693 [hep-th].
- [7] R. D. Parks, *Superconductivity*, Marcel Dekker Inc. (1969).
- [8] E. W. Carlson, V. J. Emery, S. A. Kivelson and D. Orgad, “Concepts in high temperature superconductivity,” arXiv:cond-mat/0206217.
- [9] E. Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.* **2**, 253 (1998) [arXiv:hep-th/9802150].
- [10] T. Hertog, “Towards a novel no-hair theorem for black holes,” *Phys. Rev. D* **74**, 084008 (2006) [arXiv:gr-qc/0608075].
- [11] S. S. Gubser, “Breaking an Abelian gauge symmetry near a black hole horizon,” arXiv:0801.2977 [hep-th].
- [12] P. Breitenlohner and D. Z. Freedman, “Stability In Gauged Extended Supergravity,” *Annals Phys.* **144**, 249 (1982).
- [13] I. R. Klebanov and E. Witten, “AdS/CFT correspondence and symmetry breaking,” *Nucl. Phys. B* **556**, 89 (1999) [arXiv:hep-th/9905104].
- [14] T. Hertog and G. T. Horowitz, “Towards a big crunch dual,” *JHEP* **0407**, 073 (2004) [arXiv:hep-th/0406134]; T. Hertog and G. T. Horowitz, “Holographic description of AdS cosmologies,” *JHEP* **0504**, 005 (2005) [arXiv:hep-th/0503071].
- [15] C. P. Herzog, P. Kovtun, S. Sachdev and D. T. Son, “Quantum critical transport, duality, and M-theory,” *Phys. Rev. D* **75**, 085020 (2007) [arXiv:hep-th/0701036].
- [16] D. T. Son and A. O. Starinets, “Minkowski-space correlators in AdS/CFT correspondence: Recipe and applications,” *JHEP* **0209**, 042 (2002) [arXiv:hep-th/0205051].